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An Improved Three-Stage Algorithm with Bender's Decomposition for Relative Robust Optimization Under Full Factorial Scenario Design of Data Uncertainty

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ABSTRACT

This paper presents an improved decomposition algorithm for solving two-stage relative robust optimization problems under uncertainty. The structures of the first stage and second problem are a mixed integer linear programming model (MILP) and a linear programming model (LP) respectively. Each uncertain parameter in the model can independently take its value from a finite set of real values with unknown probability distribution. This structure of parametric uncertainty is called full-factorial scenario design of data. Similar to previous work, this improved algorithm composes of three stages. The difference is that Benders' Decomposition (BD) algorithm is used to solve relaxed model in the first stage instead of the solver from CPLEX. The second and third stages are the same. The improved algorithm has been applied to solve a number of relative robust facility location problems under this structure of parametric uncertainty. All results illustrate significant improvement in computation time of the improved algorithm over existing approaches. For a problem with 3^{40} possible scenarios, an improved algorithm shows a significant reduction in computational time by 61 percents comparing with the previous three-stage algorithm without Benders' decomposition.

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1 Introduction

In this paper, we address the two-stage decision making problem under uncertainty, where the uncertainty appears in any parameter of a general MILP formulation.

$$\begin{aligned} \min \quad & \vec{c}^{\,T}\vec{x} + \vec{q}^{\,T}\vec{y} \\ \text{s.t.} \quad & \boldsymbol{W}\,\vec{y} \geq \vec{h} + \boldsymbol{T}\,\vec{x} \\ & \boldsymbol{V}\,\vec{y} = \vec{g} + \boldsymbol{S}\,\vec{x} \\ & \vec{x} \in \{0,1\}^{|\vec{x}|},\quad \vec{y} \geq \vec{0} \end{aligned}$$

The lower case letters with vector cap such as $\vec{x}$ represent vectors and the notation x_i represents the i^{th} element of the vector $\vec{x}$. The corresponding bold upper case letters such as $\boldsymbol{W}$ denote matrices and the notation W_{ij} represents the $(i, j)^{\text{th}}$ element of the matrix $\boldsymbol{W}$.

In the MILP formulation**,** let the vector $\vec{x}$ represent the first-stage binary decisions that must be made before the realization of uncertainty and let the vector $\vec{y}$ represent the second-stage continuous recourse decisions that can be made after all uncertain parameters' values are realized. It is worth emphasizing that many practical two-stage decision problems can often be represented by this mathematical formulation. For example in a facility location problem under uncertainty in a given network $G = (N, A)$, the first-stage decision $\vec{x} \in \{0,1\}^{|N|}$ represents the decision of whether or not a facility will be located at each node in the network *G*. These facility location decisions are considered as long-term decisions and must be made before the realization of uncertain parameters in the model. Once all first-stage decisions are made and all values of uncertain parameters are realized, the second-stage decisions, $\vec{y}$, are then made. In this problem, the second-stage decisions represent the flow of each material transported on each arc in the network *G*.

In this paper, we consider the problem such that each uncertain parameter in the model is restricted to independently take its value from a finite set of real values with unknown probability distribution. This captures situations where decision makers have to make the long term decision under uncertainty when there is very limited or no historical information about the decision problem and the only available information is the possible value of each uncertain parameter based on expert opinion. For example, expert opinion may give the decision makers three possible values for a given uncertain parameter based on optimistic, average, and pessimistic estimations. In this case, decision makers cannot search for the first-stage decisions with the best long run average performance, because of insufficient knowledge about the probability distribution of uncertain parameters in the model. Instead, decision makers can search for the long term decisions that perform relatively well across all possible future scenarios: robust first-stage decisions under the relative robustness definition (min-max relative regret robust solution) defined by Kouvelis and Yu (1997).

Two-stage relative robust optimization represents optimization problems where some of the model parameters are uncertain when making the first stage decisions. The criterion for the first stage decisions is to minimize the maximum relative regret between the optimal objective function value with perfect information and the resulting objective function value under the robust decisions over all possible future scenarios of the model parameters. Kouvelis and Yu (1997) summarize the state-of-the-art in the relative robust optimization and provide comprehensive discussion and motivation for applying the relative robust optimization in practice. Other works related to the robust optimization include the works by Ben-Tal et al. (2000) Mausser and Laguna (1999) Averbakh (2000, 2001), Bertsimas and Sim (2003), and Assavapokee et al. (2008, 2009).

Traditionally, solving a scenario-based extensive form model of the problem or MILP will return relative robust solution. The extensive form model is explained in section 2. The disadvantage of this method is that the problem size grows rapidly when the number of scenarios used to represent uncertainty increases. Therefore, the required computation time to find optimal solutions takes much longer. For example, a problem with 15 uncertain parameters each with 4 possible values generates over one billion scenarios. Solving the extensive form model directly obviously is not the efficient way for solving this type of robust decision problems even with the use of Benders' Decomposition (BD) technique (Benders, 1962). The scenario relaxation algorithm presented in Assavapokee et al. (2008) will certainly fail to obtain the robust solution of this considered problem. The algorithm requires solving a MILP problem for each possible scenario at the initial step. The algorithm then uses enumeration search to identify the new scenario which requires solving a LP problem for each possible scenario per iteration. This process can be quite inefficient for problems with an extremely large number of possible scenarios. For this reason, neither the extensive form model, the direct application of the BD algorithm, nor the scenario relaxation algorithm is the

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efficient methodology for solving relative robust optimization problems of this type.

Assavapokee et al. (2009) then propose a three-stage decomposition algorithm to solve the considered problem. Their algorithm is designed explicitly to efficiently handle a combinatorial sized set of possible scenarios. The algorithm sequentially solves and updates a relaxation problem until both feasibility and optimality conditions of the overall problem are satisfied. The feasibility and optimality verification steps involve the use of fractional programming techniques and bi-level programming models, which coordinates a Stackelberg game (1943) between the decision environment and decision makers. These techniques efficiently utilize the structure of uncertainty for the scenario searching procedure and are much more efficient than the enumeration search utilized by the scenario relaxation algorithm. In addition, the work by Bajalinov (2003) summarizes theoretical concepts on fractional programming. Their proposed algorithm is proven to terminate at an optimal relative robust solution if one exists in a finite number of iterations. In addition, preprocessing procedures, model transformation techniques, and problem decomposition algorithms are efficiently utilized to improve the computational tractability of the proposed algorithm.

However, for a problem with 3^{40} possible scenarios, Assavapokee et al. (2009) report the computational time at about 58 hours or about 2 and a half days to solve. This shortcoming will be improved in this paper. In this paper, eight parameters are considered, $[\vec{c}, \vec{q}, \vec{h}, \vec{g}, \boldsymbol{T}, \boldsymbol{S}, \boldsymbol{W}, \boldsymbol{V}]$ in the model. Among these parameters, we assume that the values of parameters $\vec{c}$ are known and constant during first-stage decisions but other parameter are independently chosen from a finite set of real numbers with unknown probability distribution. In the following section, we present the theoretical methodology of the improved algorithm. The performance of the improved algorithm is then demonstrated through a number of facility location problems in section 3 to show the efficiency of the improved algorithm.

2 Research Methodology

We consider the decision problem where the basic components of uncertainty are represented by a finite set of all possible scenarios of input parameters, referred as the scenario set $\bar{\Omega}$. The considered problem contains two types of decision variables. The first stage variables model binary choice decisions, which have to be made before the realization of uncertainty. The second stage decisions are continuous recourse decisions, which can be made after the realization of uncertainty. For each scenario $\omega \in \bar{\Omega}$, let vector $\vec{x}_\omega$ denote binary choice decision variables and let vector $\vec{y}_\omega$ denote continuous recourse decision variables and let $\vec{c}_\omega, \vec{q}_\omega, \vec{h}_\omega, \vec{g}_\omega, \boldsymbol{T}_\omega, \boldsymbol{S}_\omega, \boldsymbol{W}_\omega$, and $\boldsymbol{V}_\omega$ denote model parameters setting. If the realization of parameters is known to be scenario ω a priori, the optimal choice for the decision variables $\vec{x}_\omega$ and $\vec{y}_\omega$ can be obtained by solving the following model (1).

$$O^*_\omega = \min_{\vec{x}_\omega, \vec{y}_\omega} \quad \vec{c}^T_\omega \vec{x}_\omega + \vec{q}^T_\omega \vec{y}_\omega \quad \text{s.t.} \quad \begin{aligned} & \boldsymbol{W}_\omega \vec{y}_\omega - \boldsymbol{T}_\omega \vec{x}_\omega \geq \vec{h}_\omega \\ & \boldsymbol{V}_\omega \vec{y}_\omega - \boldsymbol{S}_\omega \vec{x}_\omega = \vec{g}_\omega \\ & \vec{x}_\omega \in \{0,1\}^{|\vec{x}_\omega|} \text{ and } \vec{y}_\omega \geq \vec{0} \end{aligned} \tag{1}$$

When parameters' uncertainty exists, the search for the relative robust solution comprises finding decisions, $\vec{x}$, such that the function $\max_{\omega \in \bar{\Omega}} \left((Z^*_\omega(\vec{x}) - O^*_\omega) / O^*_\omega \right)$ is minimized where

$$Z^*_\omega(\vec{x}) = \left\{ \begin{aligned} & \min_{\vec{y}_\omega} \quad \vec{q}^T_\omega \vec{y}_\omega \quad \text{s.t.} \quad \boldsymbol{W}_\omega \vec{y}_\omega \geq \vec{h}_\omega + \boldsymbol{T}_\omega \vec{x} \\ & \qquad\qquad\qquad\qquad \boldsymbol{V}_\omega \vec{y}_\omega = \vec{g}_\omega + \boldsymbol{S}_\omega \vec{x} \\ & \qquad\qquad\qquad\qquad \vec{y}_\omega \geq \vec{0} \end{aligned} \right\} + \vec{c}^T_\omega \vec{x} \quad \forall \omega \in \bar{\Omega}.$$

In the case when the scenario set $\bar{\Omega}$ is a finite set, the optimal setting of decision variables $\vec{x}$ (relative robust solution) can be obtained by solving the following model (2).

$$\begin{aligned} \min \quad \delta \quad \text{s.t.} \quad & O^*_\omega \delta \geq \vec{c}^T_\omega \vec{x} + \vec{q}^T_\omega \vec{y}_\omega - O^*_\omega && \forall \omega \in \bar{\Omega} \\ & \boldsymbol{W}_\omega \vec{y}_\omega - \boldsymbol{T}_\omega \vec{x} \geq \vec{h}_\omega && \forall \omega \in \bar{\Omega} \\ & \boldsymbol{V}_\omega \vec{y}_\omega - \boldsymbol{S}_\omega \vec{x} = \vec{g}_\omega && \forall \omega \in \bar{\Omega} \\ & \vec{y}_\omega \geq \vec{0} && \forall \omega \in \bar{\Omega} \\ & \vec{x} \in \{0,1\}^{|\vec{x}|} \end{aligned} \tag{2}$$

This model (2) is referred to in this paper as the extensive form model of the problem. If an optimal solution of the model (2) exists, the optimal setting of vector $\vec{x}$ is an optimal relative robust first stage decisions. In this work, we consider the class of decision problems such that the O^*_ω value is nonnegative $\forall \omega \in \bar{\Omega}$, which is quite common for minimum cost

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decision problems under relative robust criterion. Unfortunately, the size of the extensive form model can become unmanageably large with the size of $\bar{\Omega}$ as does the required computation time to find its optimal solution. Because of the failure of the extensive form model, the BD algorithm, and the scenario relaxation algorithm for solving a large scale problem of this type, a new decomposition algorithm which can effectively overcome limitations of these approaches is proposed in the following subsection. A key insight of the improved algorithm is to realize that, even when the size of $\bar{\Omega}$ is extremely large, it is possible to efficiently identify a smaller subset of important scenarios ($\Omega \subseteq \bar{\Omega}$) that is actually required as part of the iterative scheme to solve the overall problem with the use of bi-level programming, fractional programming, and relaxation concepts. The explanation comes from the algorithm itself. In the first two steps of the algorithm which are explained in 2.1.2, a small number of scenarios (Ω) are randomly chosen to solve for O_{ω}^{*} which is then used to update the lower bound in Step 2. In step 3, the feasibility of the problem is checked. Lastly, in Step 4, bi-level programming is used to update upper bound and identify only important scenario that will be added into Ω as shown in Assavapokee (2009). This scenario yields the maximum relative regret value possible. These steps are repeated until the gap between the lower bound and upper bound reaches the stopping criteria. Therefore, only a small number of scenarios are gradually added into the initial subset of scenarios. More details are explained in Assavapokee (2009). In addition, the improved algorithm does not require calculating the values of O_{ω}^{*} $\forall \omega \in \bar{\Omega}$ in order to solve the overall problem. These calculations can be very expensive in term of the computation time when the size of $\bar{\Omega}$ is extremely large. The algorithm only require the calculation of O_{ω}^{*} $\forall \omega \in \Omega \subseteq \bar{\Omega}$ which can be obtained quickly when the size of Ω is small.

2.1.1 Relative Robust Optimization Algorithm for Full-Factorial Scenario Design

We begin this subsection by defining some additional notations which will be extensively used in this section and the following sections of the paper. The parameters in the considered model can be combined into a random vector ξ = [$\vec{c}, \vec{q}, \vec{h}, \vec{g}, \boldsymbol{T}, \boldsymbol{S}, \boldsymbol{W}, \boldsymbol{V}$]. Because, in most cases, the values of parameters of type $\vec{c}$ are known with certainty when making the first-stage decisions, the improved algorithm only handles uncertainty for other seven types of parameters. In this work, we assume that each component of ξ (except parameters of type $\vec{c}$)

can independently take its values from a finite set of real numbers with unknown probability distribution. In other words, for each element *p* of the vector ξ, p can take any value from the set of $\{p_{(1)}, p_{(2)}, ..., p_{(\bar{p})}\}$ such that $p_{(1)} < p_{(2)} < ... < p_{(\bar{p})}$ where the notation $\bar{p}$ denotes the number of possible values for the parameter *p*. The scenario set $\bar{\Omega}$ is generated by all possible values of the parameter vector ξ. Let us define $\xi(\omega)$ as the specific setting of the parameter vector ξ under scenario $\omega \in \bar{\Omega}$ and $\Xi = \{\xi(\omega) \mid \omega \in \bar{\Omega}\}$ as the support of the unknown parameter vector ξ. As described below, we propose a three-stage decomposition algorithm for solving the relative robust optimization problem under scenario set $\bar{\Omega}$ that utilizes the efficient idea based on the following inequality where $\Omega \subseteq \bar{\Omega}$.

$$\Delta^U = \max_{\omega \in \bar{\Omega}} \left\{ \frac{Z^*_\omega(\bar{x}) - O^*_\omega}{O^*_\omega} \right\} \geq \min_{\bar{x}} \max_{\omega \in \bar{\Omega}} \left\{ \frac{Z^*_\omega(\bar{x}) - O^*_\omega}{O^*_\omega} \right\} \geq \min_{\bar{x}} \max_{\omega \in \Omega} \left\{ \frac{Z^*_\omega(\bar{x}) - O^*_\omega}{O^*_\omega} \right\} = \Delta^L$$

In the considered problem, we would like to solve the middle problem, $\min_{\bar{x}} \max_{\omega \in \bar{\Omega}} \left\{ (Z^*_\omega(\bar{x}) - O^*_\omega) / O^*_\omega \right\}$, which is intractable because $|\bar{\Omega}|$ is extremely large. Instead, we successively solve the left and right problems for Δ^U and Δ^L. The left problem provides the upper bound on the min-max relative regret value and can be solved by utilizing a reformulation as a tractable Bi-level programming model after a feasible solution $\bar{x}$ is provided. The right problem provides the lower bound on the min-max relative regret value and can be solved by utilizing the BD based on the fact that $|\Omega|$ is relatively small compared to $|\bar{\Omega}|$. The improved decomposition algorithm framework can be summarized as follows.

2.1.2 Improved Three-Stage Decomposition Algorithm Framework

***Step* 0:** (Initialization) Choose a subset $\Omega \subseteq \bar{\Omega}$ and set $\Delta^U = \infty$, and $\Delta^L = 0$. Let $\bar{x}_{opt}$ denote the incumbent solution. Determine the value of $\varepsilon \geq 0$, which is a pre-specified tolerance and proceed to Step 1.

***Step* 1:** Solve the model (1) to obtain $O^*_\omega \ \forall \omega \in \Omega$. If the model (1) is infeasible for any scenario in the scenario set Ω, the algorithm is terminated; the problem is ill-posed.

Otherwise the optimal objective function value to the model (1) for scenario ω is

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designated as O^*_ω and the algorithm proceeds to Step 2.

***Step* 2:** (Updating Lower Bound) By using the information on $O^*_\omega \ \forall \omega \in \Omega$, apply the BD algorithm explained in detail in Section 2.1 to solve the smaller version of the model (2) by considering only the scenario set Ω instead of $\bar{\Omega}$. This smaller version of the model (2) is referred to as the relaxed model $(2')$ in this paper. If the relaxed model $(2')$ is infeasible, the algorithm is terminated with the confirmation that no robust solution exists for the problem. Otherwise, set $\vec{x}_\Omega = \vec{x}^*$ which is an optimal solution from the relaxed model $(2')$ and set $\Delta^L = \max_{\omega \in \Omega} \left\{ (Z^*_\omega(\vec{x}^*) - O^*_\omega) / O^*_\omega \right\}$ which is the resulting optimal objective function value from the relaxed model $(2')$.

If $\{\Delta^U - \Delta^L\} \leq \varepsilon$, the algorithm is terminated and $\vec{x}_{opt}$ is the globally ε-optimal robust solution. Otherwise the algorithm proceeds to Step 3.

***Step* 3:** (Feasibility Check) Solve the Bi-level-1 model as described in Assavapokee et al. (2009) by using the $\vec{x}_\Omega$ information from Step 2. If the optimal objective function value of the Bi-level-1 model is nonnegative (feasible case), the algorithm proceeds to Step 4. Otherwise (infeasible case), set $\Omega \leftarrow \Omega \cup \{\omega^*_1\}$ where ω^*_1 is the infeasible scenario for $\vec{x}_\Omega$ generated by the Bi-level-1 model in this iteration and the algorithm returns to Step 1.

***Step* 4:** (Updating Upper Bound) Solve the Bi-level-2 model as described in Assavapokee et al. (2009) by using the $\vec{x}_\Omega$ information from Step 2. Let $\omega^*_2 \in \arg\max_{\omega \in \bar{\Omega}} \left\{ (Z^*_\omega(\vec{x}_\Omega) - O^*_\omega) / O^*_\omega \right\}$ and $\Delta^{U*} = \max_{\omega \in \bar{\Omega}} \left\{ (Z^*_\omega(\vec{x}_\Omega) - O^*_\omega) / O^*_\omega \right\}$ represent the results generated by the Bi-level-2 model respectively in this iteration.

If $\Delta^{U*} < \Delta^U$, then set $\vec{x}_{opt} = \vec{x}_\Omega$ and set $\Delta^U = \Delta^{U*}$.

If $\{\Delta^U - \Delta^L\} \leq \varepsilon$, the algorithm is terminated and $\vec{x}_{opt}$ is the globally ε-optimal robust solution. Otherwise, set $\Omega \leftarrow \Omega \cup \{\omega^*_2\}$ and the algorithm returns to Step 1.

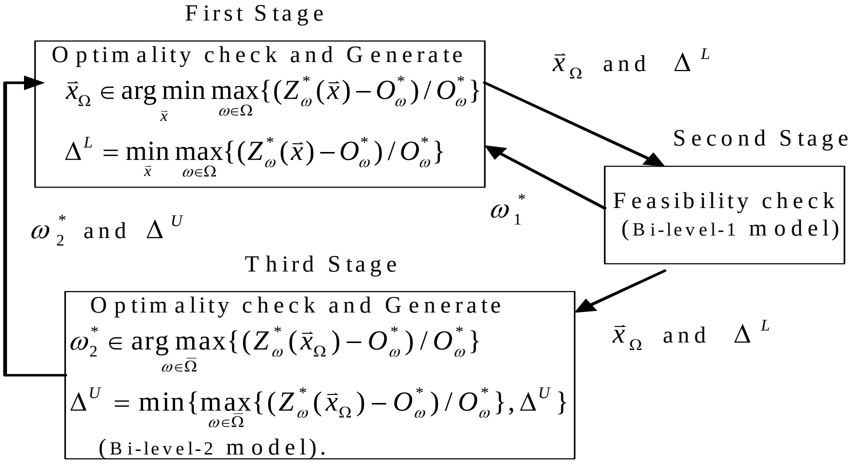

Figure 1: Schematic Structure of the Decomposition Algorithm.

We define the algorithm Steps 1 and 2 as the first stage of the algorithm and the algorithm Step 3 and Step 4 as the second stage and the third stages of the algorithm respectively. Figure 1 illustrates a schematic structure of this decomposition algorithm. Each of the three stages of the algorithm is detailed in the following subsections. Note that the first stage of the algorithm includes the BD algorithm that is used to better solve relaxed model (2'). For the second and third stages of the algorithm, all the steps are exactly the same as the ones in Assavapokee et al. (2009). The only differences are the notations which are modified so that they are more concise and simpler to understand.

2.1.3 The Improved First Stage Algorithm

The first stage algorithm will find $\bar{x}_\Omega \in \arg\min_{\bar{x}} \left\{ \max_{\omega\in\Omega}\{(Z^*_\omega(\bar{x}) - O^*_\omega)/O^*_\omega\} \right\}$ and $\Delta^L = \max_{\omega\in\Omega}\{(O^*_\omega - Z^*_\omega(\bar{x}_\Omega))/O^*_\omega\}$. In addition, it will determine whether the algorithm reaches an optimal relative robust solution. There are two main optimization models in this first stage. Model (1) is used to calculate O^*_ω for all scenarios $\omega \in \Omega \subseteq \bar{\Omega}$. For any scenario $\omega \in \Omega$, if the model (1) is infeasible, the algorithm is terminated without robust. Otherwise, once all required values of O^*_ω $\forall \omega \in \Omega$ are obtained, the relaxed model (2') is solved. The relaxed model (2') is solved using the solver from CPLEX. Due to the problem size, this step takes a lot of computational time. In this paper, we improve this step by proposing the BD techniques to solve this problem as follows. The relaxed model (2') has the structure:

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$$\min_{\vec{x},\vec{y}_\omega} (\max_{\omega\in\Omega} \{(\vec{c}_\omega^T \vec{x} + \vec{q}_\omega^T \vec{y}_\omega - O_\omega^*)/O_\omega^*\}) \quad \text{s.t.} \quad \begin{aligned} &\boldsymbol{W}_\omega \vec{y}_\omega - \boldsymbol{T}_\omega \vec{x} \geq \vec{h}_\omega && \forall \omega\in\Omega \\ &\boldsymbol{V}_\omega \vec{y}_\omega - \boldsymbol{S}_\omega \vec{x} = \vec{g}_\omega && \forall \omega\in\Omega \\ &\vec{y}_\omega \geq \vec{0} && \forall \omega\in\Omega \\ &\vec{x} \in \{0,1\}^{|\vec{x}|} \end{aligned} \qquad (2')$$

This model can be rewritten as $\min_{\vec{x}\in\{0,1\}^{|\vec{x}|}} f(\vec{x})$ where $f(\vec{x}) = \max_{\omega\in\Omega}\{(Q_\omega(\vec{x}) + c_\omega^T \vec{x} - O_\omega^*)/O_\omega^*\}$

$$Q_\omega(\vec{x}) = \min \ \vec{q}_\omega^T \vec{y}_\omega \quad \text{s.t.} \quad \begin{aligned} &\boldsymbol{W}_\omega \vec{y}_\omega \geq \vec{h}_\omega + \boldsymbol{T}_\omega \vec{x} && (\vec{\pi}_{1,\omega,\vec{x}}) \\ &\boldsymbol{V}_\omega \vec{y}_\omega = \vec{g}_\omega + \boldsymbol{S}_\omega \vec{x} && (\vec{\pi}_{2,\omega,\vec{x}}) \\ &\vec{y}_\omega \geq \vec{0} \end{aligned}$$

where the symbols in parenthesis next to the constraints denote to the corresponding dual variables. The results from the following two lemmas are used to generate the master problem and sub problems of the BD for the relaxed model $(2')$.

***Lemma* 1:** $f(\vec{x})$ is a convex function on $\vec{x}$.

***Lemma* 2:** $\left(\dfrac{\vec{c}_{\omega'}^T + (\vec{\pi}_{1,\omega',\vec{x}'}^*)^T \boldsymbol{T}_{\omega'} + (\vec{\pi}_{2,\omega',\vec{x}'}^*)^T \boldsymbol{S}_{\omega'}}{O_{\omega'}^*}\right)^T \in \partial f(\vec{x}')$ where

$\omega' \in \arg\max_{\omega\in\Omega}\{(\vec{c}_\omega^T \vec{x}' + Q_\omega(\vec{x}') - O_\omega^*)/O_\omega^*\}$, $\partial f(\vec{x}')$ is sub-differential of the function f at $\vec{x}'$ and $(\vec{\pi}_{1,\omega',\vec{x}'}^*, \vec{\pi}_{2,\omega',\vec{x}'}^*)$ is the optimal solution of the dual problem in the calculation of $Q_\omega(\vec{x})$ when $\omega = \omega'$ and $\vec{x} = \vec{x}'$. Based on the results of the Lemma 1 and 2, we summarize the BD algorithm as it applies to the relaxed model $(2')$.

Benders' Decomposition Algorithm for the Relaxed Model $(2')$:

***Step* 0:** Set lower and upper bounds $lb = -\infty$ and $ub = +\infty$ respectively. Set the iteration counter $k = 0$. Let Y^0 includes all cuts generated from all previous iterations of the improved three-stage algorithm. All these cuts are valid because the improved algorithm always add more scenarios to the set Ω and this causes the feasible region of the relaxed model $(2')$ to shrink from one iteration to the next. Let $\vec{x}^*$ denote the incumbent solution. The algorithm proceeds to Step 1.

***Step* 1:** Solve the master problem below. If the master problem is infeasible, stop and report that the relaxed model $(2')$ is infeasible.

$$lb = \min_{\theta,\vec{x}} \theta \quad \text{s.t.} \quad \theta \geq \vec{a}_i^T \vec{x} + b_i \quad \forall i = 1,2,...,k$$
$$(\theta, \vec{x}) \in Y^k$$

Otherwise, update $k \leftarrow k + 1$ and let $\vec{x}(k)$ be an optimal solution of the master problem and the algorithm proceeds to Step 2.

***Step* 2:** For each $\omega \in \Omega$, solve the following sub problem:

$$Q_\omega(\vec{x}(k)) = \min_{\vec{y}_\omega \geq \vec{0}} \vec{q}_\omega^T \vec{y}_\omega \quad \text{s.t.} \quad \boldsymbol{W}_\omega \vec{y}_\omega \geq \vec{h}_\omega + \boldsymbol{T}_\omega \vec{x}(k) \qquad (\vec{\pi}_{1,\omega,\vec{x}(k)})$$
$$\boldsymbol{V}_\omega \vec{y}_\omega = \vec{g}_\omega + \boldsymbol{S}_\omega \vec{x}(k) \qquad (\vec{\pi}_{2,\omega,\vec{x}(k)})$$

where the symbols in parenthesis next to the constraints denote to the corresponding dual variables. If the sub problem is infeasible for any scenario $\omega \in \Omega$, go to Step 5. Otherwise, using the sub problem objective values, compute the objective function value $f(\vec{x}(k)) = \left(\vec{c}_{\omega(k)}^T \vec{x}(k) + Q_{\omega(k)}(\vec{x}(k)) - O_{\omega(k)}^*\right) / O_{\omega(k)}^*$ corresponding to the current feasible solution $\vec{x}(k)$ where $\omega(k) \in \arg\max_{\omega \in \Omega}\{(\vec{c}_\omega^T \vec{x}(k) + Q_\omega(\vec{x}(k)) - O_\omega^*) / O_\omega^*\}$. If $ub > f(\vec{x}(k))$, update the upper bound $ub = f(\vec{x}(k))$ and the incumbent solution $\vec{x}^* = \vec{x}(k)$. The algorithm proceeds to Step 3.

***Step* 3:** If $ub - lb \leq \lambda$, where $\lambda \geq 0$ is a pre-specified tolerance, stop and return $\vec{x}^*$ as the optimal solution and ub as the optimal objective value; otherwise proceed to Step 4.

***Step* 4:** For the scenario $\omega(k) \in \arg\max_{\omega \in \Omega}\{(\vec{c}_\omega^T \vec{x}(k) + Q_\omega(\vec{x}(k)) - O_\omega^*) / O_\omega^*\}$, let $(\vec{\pi}^*_{1,\omega(k),\vec{x}(k)}, \vec{\pi}^*_{2,\omega(k),\vec{x}(k)})$ be the optimal dual solutions for the sub problem corresponding to $\vec{x}(k)$ and $\omega(k)$ solved in Step 2. Compute the cut coefficients

$$\vec{a}_k = \left(\frac{\vec{c}_{\omega(k)}^T + (\vec{\pi}^*_{1,\omega(k),\vec{x}(k)})^T \boldsymbol{T}_{\omega(k)} + (\vec{\pi}^*_{2,\omega(k),\vec{x}(k)})^T \boldsymbol{S}_{\omega(k)}}{O_\omega^*} \right)^T, \text{ and } b_k = -\vec{a}_k^T \vec{x}(k) + f(\vec{x}(k)), \text{ and}$$

go to Step 1.

***Step* 5:** Let $\hat{\omega} \in \Omega$ be a scenario such that the sub problem is infeasible. Solve the following optimization problem where $\vec{0}$ and $\vec{1}$ represent the vector with all elements equal to zero and one respectively.

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$$\max_{\vec{v}_1,\vec{v}_2}(\vec{h}_{\hat{\omega}}+\boldsymbol{T}_{\hat{\omega}}\vec{x}(k))^T\vec{v}_1+(g_{\hat{\omega}}+\boldsymbol{S}_{\hat{\omega}}\vec{x}(k))^T\vec{v}_2 \quad \text{s.t.} \quad \boldsymbol{W}_{\hat{\omega}}^T\vec{v}_1+\boldsymbol{V}_{\hat{\omega}}^T\vec{v}_2\le\vec{0}$$

$$\vec{0}\le\vec{v}_1\le\vec{1}, \quad -\vec{1}\le\vec{v}_2\le\vec{1}$$

Let $\vec{v}_1^*$ and $\vec{v}_2^*$ be the optimal solution of this optimization problem. Set $k \leftarrow k - 1$ and $Y^k \leftarrow Y^k \cap \{\vec{x} \mid (\vec{h}_{\hat{\omega}}+\boldsymbol{T}_{\hat{\omega}}\vec{x})^T\vec{v}_1^*+(\vec{g}_{\hat{\omega}}+\boldsymbol{S}_{\hat{\omega}}\vec{x})^T\vec{v}_2^*\le 0\}$ and go to Step 1.

If the relaxed model (2') returns an infeasible solution, the algorithm is terminated without robust solution to the problem. Otherwise, its results are the candidate robust decisions, $\vec{x}_\Omega=\vec{x}^*$, and the lower bound on min-max relative regret value, $\Delta^L=\max_{\omega\in\Omega}\{(Z_\omega^*(\vec{x}^*)-O_\omega^*)/O_\omega^*\}$ obtained from the relaxed model (2'). The optimality condition is satisfied when $\Delta^U-\Delta^L\le\varepsilon$, where $\varepsilon\ge 0$ is pre-specified tolerance. If the optimality condition is satisfied, the algorithm is stopped with the solution $\vec{x}_{opt}$ as the ε-optimal relative robust solution. Otherwise the solution $\vec{x}_\Omega$ and the value of Δ^L are forwarded to the second stage to find better solution. For second and third stages, the solution procedures are the same as Assavapokee et al. (2009).

3 Numerical Experiments

In this section, we describe numerical experiments using the improved algorithm for solving a number of two-stage facility location problems under uncertainty. The considered supply chain composes of suppliers, factories, warehouses and markets. The decisions on the location, capacity allocation, and transportation are determined in order to minimize the overall supply chain cost. The model requires the following notations, parameters and decision variables:

m	Number of markets	c_{1hi}	Cost of shipping one unit from supplier h to factory i
n	Number of potential factory locations	c_{2ie}	Cost of shipping one unit from factory i to warehouse e
l	Number of suppliers	c_{3ej}	Cost of shipping one unit from warehouse e to market j
t	Number of potential warehouse locations	p_j	Penalty cost per unit of unsatisfied demand at market j
D_j	Annual demand from customer j	z_e	= 1 if warehouse is opened at site e; 0 otherwise
K_i	Potential capacity of factory site i	x_{2ie}	Transportation quantity from plant i to warehouse e
S_h	Supply capacity at supplier h	s_j	Quantity of unsatisfied Demand at market j
W_e	Potential warehouse capacity at site e	y_i	= 1 if plant is opened at site i; : = 0 otherwise

f_{1i}	Fixed cost of locating a plant at site i	x_{1hi}	Transportation quantity from supplier h to plant i
f_{2e}	Fixed cost of locating a warehouse at site e	x_{3ej}	Transportation quantity from warehouse e to market j

We assume that there can be many warehouses and factories to satisfy every player the entire supply chain and one unit of input from a supply source corresponds to one unit of the finished product. There are also the capacity limits for all factories and warehouses. Also, there is a linear penalty cost for each unit of unsatisfied demands. For the deterministic case, the overall problem can be modeled as the MILP problem presented in the following model. When some parameters in the model are uncertain, the goal becomes to identify robust factory and warehouse locations under relative robustness definition. Transportation decisions are treated as recourse decisions which will be made after the realization of uncertainty.

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} f_{1i} y_i + \sum_{e=1}^{t} f_{2e} z_e + \sum_{h=1}^{l}\sum_{i=1}^{n} c_{1hi} x_{1hi} + \sum_{i=1}^{n}\sum_{e=1}^{t} c_{2ie} x_{2ie} + \sum_{e=1}^{t}\sum_{j=1}^{m} c_{3ej} x_{3ej} + \sum_{j=1}^{m} p_j s_j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{1hi} \le S_h \quad \forall h \in \{1,\dots,l\}, \qquad \sum_{h=1}^{l} x_{1hi} - \sum_{e=1}^{t} x_{2ie} = 0 \quad \forall i \in \{1,\dots,n\} \\
& \sum_{e=1}^{t} x_{2ie} \le K_i y_i \quad \forall i \in \{1,\dots,n\}, \qquad \sum_{i=1}^{n} x_{2ie} - \sum_{j=1}^{m} x_{3ej} = 0 \quad \forall e \in \{1,\dots,t\} \\
& \sum_{j=1}^{m} x_{3ej} \le W_e z_e \quad \forall e \in \{1,\dots,t\}, \qquad \sum_{e=1}^{t} x_{3ej} + s_j = D_j \quad \forall j \in \{1,\dots,m\} \\
& x_{1hi} \ge 0 \;\; \forall h\, \forall i,\; x_{2ie} \ge 0 \;\; \forall i\, \forall e,\; x_{3ej} \ge 0 \;\; \forall e\, \forall j,\; s_j \ge 0 \;\; \forall j,\; y_i \in \{0,1\} \;\; \forall i, \text{ and } z_e \in \{0,1\} \;\; \forall e
\end{aligned}$$

We apply the improved algorithm to 25 different experimental settings of the robust facility location problems. Each experimental setting in this case study contains different sets of uncertain parameters and different sets of possible locations which result in different number of possible scenarios. The number of possible scenarios in this case study varies from 64 up to 3^{40} scenarios. The key uncertain parameters in these problems are the supply quantity at the supplier, the potential capacity at the factory, the potential capacity at the warehouse, and the unit penalty cost for not meeting demand at the market. Let us define notations *l'*, *n'*, *t'*, and *m'* to represent the number of suppliers, factories, warehouses, and markets with uncertain parameters respectively in the problem. It is assumed that each uncertain parameter in the model can independently take its values from *r* possible real values. The following Table 1 describes these twenty five settings of the case study. Table 2 summarizes information on approximated parameters' values associated with case 25. A variable transportation cost of \$0.01 per unit per mile is assumed in this case. The distance between each pair of locations is calculated based on the latitude and longitude information. It is assumed that each uncertain parameter in this case can take its values from 80%, 100%, and 120% of its approximated value reported in Table 2. These data are the same as the data of the large problem solved by Assavapokee et al. (2009).

Table 1: Twenty Five Settings of Numerical Problems in the Case Study.

Problem	l	n	t	m	l'	n'	T'	m'	r	$\lvert\bar{\Omega}\rvert$	Size of Extensive Form Model		
											#Constraints	#Continuous Variables	#Binary Variables
1	8	8	8	8	0	2	2	2	2	64	3.14×10^{3}	1.13×10^{4}	16
2	8	8	8	8	0	2	2	4	2	256	1.25×10^{4}	4.51×10^{4}	16
3	8	8	8	8	0	2	4	2	2	256	1.25×10^{4}	4.51×10^{4}	16
4	8	8	8	8	0	2	6	2	2	1024	5.02×10^{4}	1.80×10^{5}	16
5	8	8	8	8	0	2	2	6	2	1024	5.02×10^{4}	1.80×10^{5}	16
6	8	8	8	8	0	2	4	4	2	1024	5.02×10^{4}	1.80×10^{5}	16
7	8	8	8	8	0	2	4	6	2	4096	2.01×10^{5}	7.21×10^{5}	16
8	8	8	8	8	0	2	6	4	2	4096	2.01×10^{5}	7.21×10^{5}	16
9	8	8	8	8	0	2	6	6	2	2^{14}	8.03×10^{5}	2.88×10^{6}	16
10	8	8	8	8	0	6	6	6	2	2^{18}	1.28×10^{7}	4.61×10^{7}	16
11	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
12	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
13	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
14	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
15	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
16	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
17	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
18	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
19	6	6	6	6	6	6	6	6	3	3^{24}	1.04×10^{13}	2.71×10^{13}	12
20	8	8	8	8	8	8	8	8	3	3^{32}	9.08×10^{16}	3.26×10^{17}	16
21	8	8	8	8	8	8	8	8	3	3^{32}	9.08×10^{16}	3.26×10^{17}	16
22	8	8	8	8	8	8	8	8	3	3^{32}	9.08×10^{16}	3.26×10^{17}	16
23	8	8	8	8	8	8	8	8	3	3^{32}	9.08×10^{16}	3.26×10^{17}	16
24	8	8	8	8	8	8	8	8	3	3^{32}	9.08×10^{16}	3.26×10^{17}	16
25	10	10	10	10	10	10	10	10	3	3^{40}	7.42×10^{20}	3.40×10^{21}	20

Table 2: Approximated Parameters' Information of the Case 25.

Supplier (h)	S_h	Factory (i)	f_{1i}	K_i	Warehouse (e)	f_{2e}	W_e	Market (j)	D_j	p_j
San Diego	2500	Seattle	75000	2800	Sacramento	37500	2500	Portland	800	125
Denver	3000	San Franc.	75000	2400	Oklahoma C.	37500	2600	LA	1050	140
Kansas C.	2000	Salt Lake	75000	2500	Lincoln	37500	2500	Phoenix	600	120
El Paso	1000	Wilmingt.	60000	2150	Nashville	40500	3000	Houston	1800	150
Cincinnati	800	Dallas	75000	2700	Cleveland	37500	2600	Miami	1500	125
Boise	500	Minneapo.	60000	2100	Fort Worth	36000	2100	New York	1250	130
Austin	1100	Detroit	81000	3000	Eugene	37500	2300	St. Louis	1050	120
Charlotte	2000	Tampa	39000	1400	Santa Fe	34500	1700	Chicago	1750	140
Huntsville	700	Pittsburgh	75000	2500	Jacksonv.	37500	2400	Philadel.	750	125
Bismarck	600	Des Moin.	45000	1600	Boston	37500	2600	Atlanta	1850	155

Table 3: Performance Comparison between Proposed Algorithm and Traditional Methods.

Problem	Solution Time (sec.)			Proposed Algorithm							
				Initial Scenarios Setup 1				Initial Scenarios Setup 2			
	EFM	BEFM	SR	Stage1 Time (sec.)	Stage3 Time (sec.)	Total Time (sec.)	#Iteration (|Ω|)	Stage1 Time (sec.)	Stage3 Time (sec.)	Total Time (sec.)	#Iteration (|Ω|)
1	460	154.26	35	18.64	6.65	25.3	4 (5)	6.15	2.14	8.3	1 (9)
2	6378	1025.64	83	26.53	5.96	32.5	4 (5)	6.98	2.51	9.5	1 (9)
3	10790	1242.53	104	29.82	8.87	38.7	6 (7)	6.40	2.29	8.7	1 (9)
4	55481	3652.87	511	38.61	8.58	47.2	8 (9)	7.25	2.84	10.1	1 (9)
5	65269	4058.65	379	28.30	6.49	34.8	4 (5)	9.93	2.66	12.6	2 (10)
6	62832	4123.36	450	31.74	10.65	42.4	6 (7)	12.02	3.17	15.2	3 (11)
7	--	15254	1827	45.74	8.55	54.3	9 (10)	9.02	2.77	11.8	2 (10)
8	--	17329	2123	45.96	10.24	56.2	9 (10)	14.63	2.56	17.2	3 (11)
9	--	79602	8895	40.38	9.01	49.4	8 (9)	17.01	2.59	19.6	3 (11)
10	--	--	184868	98.52	14.48	113	15 (16)	42.33	9.76	52.1	5 (13)
11	--	--	--	78.80	50.43	129.23	13 (14)	19.46	6.42	25.88	2 (18)
12	--	--	--	106.52	35.62	142.14	16 (17)	138.83	32.99	171.82	13 (29)
13	--	--	--	64.76	47.84	112.60	13 (14)	102.1	40.04	142.14	11 (27)
14	--	--	--	48.30	43.09	91.39	9 (10)	62.37	21.32	83.69	7 (23)
15	--	--	--	72.24	41.98	114.22	14 (15)	32.7	14.19	46.89	3 (19)
16	--	--	--	246.84	128.04	374.88	28 (29)	150.52	63.01	213.53	14 (30)
17	--	--	--	129.56	17853	17982	18 (19)	118.88	10546	10665	13 (29)
18	--	--	--	82.27	41.10	123.37	14 (15)	66.31	6.26	72.57	3 (19)
19	--	--	--	168.50	6908.16	7076.66	20 (21)	79.21	12263	12342	7 (23)
20	--	--	--	459.74	1289.83	1749.57	23 (24)	219.13	766.63	985.76	8 (24)
21	--	--	--	1097.64	67162	68260	39 (40)	634.44	66510	67144	20 (36)
22	--	--	--	540.77	20151	20692	26 (27)	315.15	17874	18189	16 (32)
23	--	--	--	1450.48	420.33	1870.22	48 (49)	538.36	511.07	1049.43	16 (32)
24	--	--	--	1419.59	36800	38219	41 (42)	527.08	40180	40707	18 (34)
25	--	--	--	3032.8	120231	123264	84 (85)	1882.54	80709	82592	57 (73)

All case study settings are solved by the improved algorithm with ε = 0, the scenario relaxation algorithm (SR) with one initial scenario, extensive form model (EFM), and Benders' Decomposition (BEFM) on a Windows XP-based Pentium(R) 4 CPU 3.60GHz personal computer with 2.00 GB RAM using a C++ program and CPLEX Concert Technology API for the optimization process. MS-Access is used for input and output database. In this case study, we apply the improved algorithm to these 25 experimental problems by using two different setups of initial scenarios. For the first setup, the initial scenario set consists of only one scenario where all parameters are set at their approximated value. For example, the initial scenario for the case 25 is the scenario where all parameters take their values from associated values reported in the Table 2. For the second setup, the

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initial scenario set consists of all possible combinations of upper and lower bounds for each main type of uncertain parameters. For example, there are four main types of uncertain parameters (supply quantity, factory capacity, warehouse capacity, and penalty cost) in the case 25. In this case, the initial scenario set of the problem will consist of $2^4 = 16$ scenarios for the second setup. Table 3 illustrates the computation time (in seconds) and performance comparison among these methodologies over all 25 settings. If the algorithm fails to obtain an optimal robust solution within 52 hours or fails to solve the problem due to insufficiency of memory, the computation time of "--" is reported in the table. Because the problems considered in this case study are always feasible for any setting of location decisions, the Stage2 of the improved algorithm can be omitted.

All results from these experimental runs illustrate significant improvements in computation time of the improved algorithm over the scenario relaxation algorithm and the extensive form model both with and without BD. These results demonstrate the promising capability of the improved algorithm for solving practical relative robust optimization problems under full factorial scenario design of data uncertainty with extremely large number of possible scenarios. In addition, these numerical results illustrate the impact of the quality of the initial scenarios setup on the required computation time of the improved algorithm. Decision makers are highly recommended to perform thorough analysis of the problem in order to construct the good initial set of scenarios before applying the improved algorithm. For case 25 which has 3^{40} possible scenarios, the algorithm proposed by Assavapokee et al. (2009) takes 57 iterations or 58 hours 27 minutes and 34 seconds to complete. Our improved algorithm with second setup also takes the same number of iterations but finishes in 22 hours 56 minutes and 32 seconds only. The computational time is reduced by about 61 percent. This result shows that our improved algorithm is more efficient to solve this type of problem.

4 Conclusion

This paper proposes an improved relative robust optimization algorithm that can account for the uncertainty in model parameter values of MILP problems when each uncertain parameter in the model can independently take its value from a finite set of real numbers with unknown joint probability distribution. This type of parametric uncertainty is referred to as a full-factorial scenario design. The algorithm consists of three stages to solve the overall optimization problem efficiently. Using the pre-processing steps, decomposition techniques, and problem transformation procedures, the improved algorithm has shown computational improvement. The main contribution of this paper is the addition of BD algorithm in stage one which replaces the default algorithm from CPLEX as done by Assavapokee et al. (2009). This algorithm help improve the computation time so that the larger problem can be solved quicker. We also show that the algorithm will either terminate at an optimal relative robust solution or identify the infeasibility in a finite number of iterations. To verify the improvement of BD algorithm, twenty five case studies of robust facility location problems under uncertainty are solved using the improved algorithm. These case studies are originated by Assavapokee et al. (2009). All results illustrate better performance of the improved algorithm for solving the relative robust optimization problems of this type over the traditional methodologies. When the solution computation effort is compared with algorithm of Assavapokee et al. (2009) without BD algorithm, the improved algorithm can reduced the computational time by 61 percents for the problem with 3^{40} possible scenarios. This improvement enables the researcher to solve a large problem faster.

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